P. S. Since the writing of the above, I have been inform'd, that, in two or three houſes, ſinging birds, which were at the time rooſting in their cages, were thrown off their perches by the ſuddenneſs of the ſhock.

LXXVI. *An Inveſtigation of ſome Theorems which ſuggeſt ſome remarkable Properties of the Circle, and are of Uſe in reſolving Fractions, whoſe Denominators are certain Multinomials, into more ſimple ones. By Mr.* John Landen.

Read May 2, 1754.

THAT the principal theorems, below inveſtigated, will be of conſiderable uſe in the doctrine of fluxions, by rendering, in many caſes, the buſineſs of computing fluents more eaſy, will, on peruſal, be obvious to every one acquainted with that branch of ſcience. Therefore, as the facilitating computations in that doctrine (which affords us wonderful aſſiſtance in many phyſical enquiries) may be a means of extending our knowlege in philoſophy; it is preſum'd, that this paper will not be thought unworthy the notice of the Royal Society.

I.

Suppoſing $\frac{n\dot{x}}{\sqrt{x^2-1}}=\frac{\dot{y}}{\sqrt{y^2-1}}$, where $\dot{x}$ and $\dot{y}$ denote the fluxions of the variable quantities x and y re-

reſpectively, and n an invariable quantity; it is propos'd to find, in terms of y and z, the equation of which z is a root, and $z^2 - 2xz + 1 = 0$, a diviſor.

Taking the fluents of the given fluxionary equation, we have, ſuppoſing $x = 1$ when y is $= 1$, hyp. log. of $\overline{x + \sqrt{x^2 - 1}}^{\,n}$ = hyp. log. of $y + \sqrt{y^2 - 1}$, or $\overline{x + \sqrt{x^2 - 1}}^{\,n} = y + \sqrt{y^2 - 1}$: Whence, by ſubſtituting for x its value $\frac{z^2 + 1}{2z}$ (found by the equation $z^2 - 2xz + 1 = 0$), we have $z^n = y + \sqrt{y^2 - 1}$: Therefore $z^n - y$ is $= \sqrt{y^2 - 1}$; and, ſquaring both ſides, $z^{2n} - 2yz^n + y^2 = y^2 - 1$. Conſequently $z^{2n} - 2yz^n + 1$ is $= 0$; which, ſuppoſing n a poſitive integer, is the equation ſought.

Now it is obvious, n being ſuch an integer, that this equation will have as many trinomial diviſors, of the form $z^2 - 2xz + 1$, as there are values of x correſponding to a given value of y: Which values of x, when y is not greater than 1, nor leſs than -1 (the only caſe I purpoſe to conſider), will not be readily obtain'd from the equation $\overline{x + \sqrt{x^2 - 1}}^{\,n} = y + \sqrt{y^2 - 1}$ found above: But, if we multiply the given fluxionary equation by $\frac{1}{\sqrt{-1}}$, we get $\frac{n\dot{x}}{\sqrt{1 - x^2}} = \frac{\dot{y}}{\sqrt{1 - y^2}}$; of which the equation of the fluents is $n \times$ circ. arc rad. 1. coſine x = circ. arc rad. 1. coſine y; where x is $= 1$ when y is $= 1$, agreeable to the ſuppoſition we made above when we took the fluents of the given fluxionary equation by logarithms. Therefore if A be put for the leaſt arc whoſe coſine is y, and C for the whole circumference, radius being 1; y being the coſine of A, $A+C$, $A+2C$, $A+3C$, &c. x will be the coſine of $\frac{A}{n}$, $\frac{A+C}{n}$, $\frac{A+2C}{n}$, &c. . . . . to $\frac{A+\overline{n-1}\times C}{n}$.

Conſequently, expreſſing the laſt-mention'd coſines, or the ſeveral values of x, by p, q, r, s, &c. $z^{2n} - 2yz^{n} + 1$ will be $= \overline{z^2 - 2pz + 1} \times \overline{z^2 - 2qz + 1} \times \overline{z^2 - 2rz + 1}$, &c. ($n$), when n is a poſitive integer (as we ſhall always ſuppoſe it to be), let z be what it will.

Hence may be eaſily deduc'd a demonſtration of that remarkable property of the circle firſt diſcover'd by Mr. Cotes: But as that property has already been demonſtrated by ſeveral mathematicians, I ſhall omit taking any farther notice of it, and proceed in the inveſtigation of ſome other uſeful theorems which I do not find have ever yet been publiſh'd.

II.

If y be $= 1$; then, A being $= 0$; p, q, r, &c. will be the coſines of $\frac{0}{n}$, $\frac{C}{n}$, $\frac{2C}{n}$, $\frac{3C}{n}$, &c. (n) reſpectively: Therefore p will be $= 1$; and, if n be an even number, one of the coſines q, r, s, &c. will be $= -1$, one of the arcs $\frac{C}{n}$, $\frac{2C}{n}$, $\frac{3C}{n}$, &c. being then $= \frac{C}{2}$.

III.

III.

If y be $= -1$; then, A being $= \frac{C}{2}$; p, q, r, s, &c. will be the coſines of $\frac{C}{2n}$, $\frac{3C}{2n}$, $\frac{5C}{2n}$, &c. (n) reſpectively: Therefore, if n be an odd number, one of thoſe arcs will be $\frac{C}{2}$, whoſe coſine is -1.

IV.

If in the equations $z^{2n} - 2yz^n + 1 = 0$, and $z^2 - 2xz + 1 = 0$, we ſubſtitute $v-1$ for z, they become $\overline{v-1}^{2n} - 2y \times \overline{v-1}^{n} + 1 = 0$, and $\overline{v-1}^{2} - 2x \times \overline{v-1} + 1 = v^2 - \overline{2+2x} \times v + 2 + 2x = 0$. Conſequently

$$\left.\begin{aligned} &v^{2n} - 2nv^{2n-1} + \ldots\ldots + 2n \times \frac{2n-1}{2} v^2 - 2nv + 1 \\ &\qquad\qquad\qquad \ldots\ldots \mp 2yn \times \frac{n-1}{2} v^2 \pm 2ynv \mp 2y \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 1 \end{aligned}\right\} =$$

$$\overline{v^2 - \overline{2+2p} \times v + 2 + 2p} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q} \times$$
$$\overline{v^2 - \overline{2+2r} \times v + 2 + 2r} \times \text{\&c.}\ (n);$$

where, of the two ſigns prefix'd to the terms where y is a factor, the upper or lower takes place, according as n is an even or an odd number. Whence, by the nature of equations, it follows, that $\overline{2+2p} \times \overline{2+2q} \times \overline{2+2r}$, &c. is $= 2 \mp 2y$. But this equation vaniſhing when y is $= 1$ and n an even number, or when y is $= -1$ and

and n an odd number, it will be proper to conſider thoſe two caſes more particularly.

V.

Firſt, Let us ſuppoſe $y = 1$, and n an even number: Then p being $= 1$, and one of the other coſines q, r, s, &c. $= -1$ (*Art.* II.); we ſhall have

$$v^{2n} - 2nv^{2n-1} + \ldots + n^2v^2 = \overline{v^2+0} \times \overline{v^2 - 4v + 4} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q} \times \overline{v^2 - \overline{2+2r} \times v + 2 + 2r},$$

&c. Therefore dividing by v^2,

$$v^{2n-2} - 2nv^{2n-3} + \ldots + n^2 = \overline{v^2 - 4v + 4} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q} \times \overline{v^2 - \overline{2+2r} \times v + 2 + 2r},$$

&c. that factor in which the value of the coſine q, or r, &c. is -1, being expung'd.

Conſequently n^2 is $= 4 \times \overline{2+2q} \times \overline{2+2r} \times \overline{2+2s}$, &c. when the factor, whoſe value is nothing, is expung'd.

VI.

Let us now ſuppoſe $y = -1$, and n an odd number: Then one of the coſines p, q, r, &c. being $= -1$ (*Art.* III.),

$v^{2n} - 2nv^{2n-1} + \ldots + n^2v^2$ will be $= \overline{v^2+0} \times \overline{v^2 - \overline{2+2p} \times v + 2 + 2p} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q}$, &c. Therefore, dividing by v^2,

$v^{2n-2} - 2nv^{2n-3} + \ldots + n^2$ will be $= \overline{v^2 - \overline{2+2p} \times v + 2 + 2p} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q}$, &c. and conſequently $n^2 = \overline{2+2p} \times \overline{2+2q} \times \overline{2+2r}$, &c. when the factor, whoſe value is nothing, is expung'd.

VII.

Subſtituting in the equations $z^{2n} - 2yz^{n} + 1 = 0$, and $z^{2} - 2xz + 1 = 0$, $\frac{a+\omega}{a-\omega}$ inſtead of z, we have

$$\left.\overline{\frac{a+\omega}{a-\omega}}\right|^{2n} - 2y \times \left.\overline{\frac{a+\omega}{a-\omega}}\right|^{n} + 1$$

$$= \frac{\overline{a+\omega}^{\,2n} - 2y \times \overline{a+\omega}^{\,n} \times \overline{a-\omega}^{\,n} + \overline{a-\omega}^{\,2n}}{\overline{a-\omega}^{\,2n}} = 0, \text{ and}$$

$$\left.\overline{\frac{a+\omega}{a-\omega}}\right|^{2} - 2x \times \frac{a+\omega}{a-\omega} + 1$$

$$= \frac{\overline{a+\omega}^{\,2} - 2x \times \overline{a+\omega} \times \overline{a-\omega} + \overline{a-\omega}^{\,2}}{\overline{a-\omega}^{\,2}}$$

$$= \frac{\overline{2+2x} \times \overline{\omega^{2} + \frac{1-x}{1+x}a^{2}}}{\overline{a-\omega}^{\,2}} = 0.$$ Confequently

$\overline{a+\omega}^{\,2n} - 2y \times \overline{a+\omega}^{\,n} \times \overline{a-\omega}^{\,n} + \overline{a-\omega}^{\,2n}$ will be $=$ $\overline{2+2p} \times \overline{2+2q} \times \overline{2+2r}$, &c. $\times \overline{\omega^{2} + \frac{1-p}{1+p}a^{2}} \times \overline{\omega^{2} + \frac{1-q}{1+q}a^{2}} \times \overline{\omega^{2} + \frac{1-r}{1+r}a^{2}}$, &c.

But, by *Art.* IV. $\overline{2+2p} \times \overline{2+2q} \times \overline{2+2r}$, &c. is $= 2 \mp 2y$, the upper or lower of the two ſigns prefix'd to y taking place according as n is an even or an odd number.

Therefore $\overline{a+\omega}^{2n} - 2y\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{n}$ is $=$ $\overline{2+2y\times\omega^2+\frac{1-p}{1+p}a^2}\times\overline{\omega^2+\frac{1-q}{1+q}a^2}\times\overline{\omega^2+\frac{1-r}{1+r}a^2}$, &c.

Now p being the coſine of any number of degrees, radius being 1, $\frac{1-p}{1+p}a^2$ will be the ſquare of the tangent of half ſo many degrees, radius being a: Therefore, denoting that tangent by b; and the tangents of half the arcs deſcrib'd with the radius a, whoſe coſines, when the radius is 1, are q, r, s, &c. being denoted by c, d, e, &c. reſpectively; we have $\overline{a+\omega}^{2n} - 2y\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{2n} = \overline{2\mp2y\times\omega^2+b^2}\times\overline{\omega^2+c^2}\times\overline{\omega^2+d^2}$, &c. But when y is $=1$, and n an even number; or $y=-1$, and n an odd number; $2\mp2y$ being $=0$; nothing can be determin'd from that equation: Wherefore, in thoſe caſes, recourſe muſt be had to what is done above.

VIII.

Let us ſuppoſe $y=1$, and n an even number: Then the equation $\overline{a+\omega}^{2n} - 2y\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n} + \overline{a-\omega}^{2n} = \overline{2+2p}\times\overline{2+2q}\times\overline{2+2r}$, &c. $\overline{\times\omega^2+\frac{1-p}{1+p}a^2}\times\overline{\omega^2+\frac{1-q}{1+q}a^2}\times\overline{\omega^2+\frac{1-r}{1+r}a^2}$, &c. becomes

becomes $\overline{a+\omega}^{2n} - 2\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{2n} = 4\times\overline{2+2q}\times\overline{2+2r}$, &c. $\omega^2\times\overline{\omega^2+\frac{1-q}{1+q}a^2}\times\overline{\omega^2+\frac{1-r}{1+r}a^2}$, &c. p being $=1$ (*Art.* II.) and $\frac{1-p}{1+p}a^2\ (=b^2)=0$. Moreover, one of the other coſines q, r, s, &c. being $=-1$ (*Art.* II.), ſome one of the factors $2+2q$, $2+2r$, $2+2s$, &c. will vaniſh: Which factor being expung'd from the product $4\times\overline{2+2q}\times\overline{2+2r}$, &c. and reſtor'd to the diviſor $\omega^2+\frac{1-q}{1+q}a^2$, or $\omega^2+\frac{1-r}{1+r}a^2$, &c. from which it was taken, that diviſor will become $4a^2$; and the product $4\times\overline{2+2q}\times\overline{2+2r}$, &c. will then (by *Art.* V.) be $=n^2$.

Conſequently $\overline{a+\omega}^{2n}-2\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{2n}$, will be $=n^2\times\omega^2\times 4a^2\times\overline{\omega^2+c^2}\times\overline{\omega^2+d^2}$, &c. where the factor $4a^2$ takes place inſtead of ω^2+ ſq. of the tang. of 90°.

If y be $=1$, and n an odd number, p will be $=1$, and $b=0$; but no one of the coſines q, r, s, &c. will be $=-1$, as when n is an even number. Therefore, in this caſe, the equation $\overline{a+\omega}^{2n}-2y\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{2n}=\overline{2\mp 2y}\times\overline{\omega^2+b^2}\times\overline{\omega^2+c^2}$, &c. becomes $\overline{a+\omega}^{2n}-2\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{n}=4\times\omega^2\times\overline{\omega^2+c^2}\times\overline{\omega^2+d^2}$, &c.

IX.

By taking the ſquare root of $\overline{a+\omega}^{2n} - 2 \times \overline{a+\omega}^{n} \times \overline{a-\omega}^{n} + \overline{a-\omega}^{2n}$, and of its two values juſt now found, we have, when n is an even number, $\overline{a+\omega}^{n} - \overline{a-\omega}^{n} = 2an\omega \times \sqrt{\omega^2+c^2} \times \sqrt{\omega^2+d^2}$, &c. $2a$ taking place inſtead of $\sqrt{\omega^2 + \text{ſq. of the tang. of } 90^\circ}$.

And, when n is an odd number, $\overline{a+\omega}^{n} - \overline{a-\omega}^{n} = 2\omega \times \sqrt{\omega^2+c^2} \times \sqrt{\omega^2+d^2}$, &c. Whence the following conſtruction is inferr'd.

X.

Deſcribe about the centre C (*Plate* XX. *fig.* 1. and 2.), with the radius a, the circle $P A' A'' A'''$, &c.; draw the diameter $P C Q$, and the tangent $B''' P B^5$; divide the ſemicircumference $P A' Q$ into as many equal parts $P A'$, $A' A''$, $A'' A'''$, &c. as there are units in the integer n; draw the ſecants $C A' B'$, $C A'' B''$, &c. and, taking on $C Q$ any point O, draw $K''' O K^5$ parallel to $B''' P B^5$; likewiſe draw $B' K'$, $B'' K''$, $B''' K'''$, &c. parallel to $P Q$; and call $C O$, ω.

Then will q be the coſine of twice the angle $P C A'$, r the coſine of twice $P C A''$, s the coſine of twice $P C A'''$, &c. if the radius be 1.

Therefore $P B' = O K'$ will be $= c$, $P B'' = O K'' = d$, &c. and $C K' = \sqrt{\omega^2+c^2}$, $C K'' = \sqrt{\omega^2+d^2}$, &c. Conſequently $O P^n - O Q^n$ being $= \overline{a+\omega}^{n}$

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$-\overline{a-\omega}^{n}$, and $n \times PQ \times CO \times CK' \times CK''$, &c. $= 2an\omega \times \sqrt{\omega^2 + c^2} \times \sqrt{\omega^2 + d^2}$, &c. when n is an even number; $OP^n - OQ^n$ will then be $= n \times PQ \times CO \times CK' \times CK''$, &c. where the diameter PQ takes place inſtead of the infinite quantity $C\overset{\frac{n}{2}}{K}$.

But if n be an odd number, $OP^n - OQ^n$ will be $= 2 \times CO \times CK' \times CK'' \times CK'''$, &c.

XI.

It is evident that, of the factors CK', CK'', CK''', &c. the firſt and laſt, the ſecond and laſt but one, *&c.* are reſpectively equal to each other. Therefore, omitting the ſquares of the factors below PQ, and the ſquares of their values,
$OP^n - OQ^n$ is $= n \times PQ \times CO \times CK'^2 \times CK''^2 \times CK'''^2$, &c. and $\overline{a+\omega}^{n} - \overline{a-\omega}^{n} = 2an\omega \times \overline{\omega^2 + c^2} \times \overline{\omega^2 + d^2}$, &c. when n is an even number; or
$OP^n - OQ^n$ is $= 2 \times CO \times CK'^2 \times CK''^2 \times CK'''^2$, &c. and $\overline{a+\omega}^{n} - \overline{a-\omega}^{n} = 2\omega \times \overline{\omega^2 + c^2} \times \overline{\omega^2 + d^2}$, &c. when n is an odd number.

XII.

If we ſuppoſe $y = -1$, and n an odd number, it will appear, by proceeding much in the ſame manner as in *Art.* VIII. that $\overline{a+\omega}^{2n} + 2 \times \overline{a+\omega}^{n} \times \overline{a-\omega}^{n} + \overline{a-\omega}^{2n}$ is $= n^2 \times 4a^2 \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2} \times \overline{\omega^2 + d^2}$, &c.

&c. where the factor $4a^2$ takes place inſtead of $\omega^2 +$ ſq. of the tang. of 90°.

If y be $= -1$, and n an even number, $\overline{a+\omega}^{2n} + 2 \times \overline{a+\omega}^{n} \times \overline{a-\omega}^{n} + \overline{a-\omega}^{2n}$ is $= 4 \times \overline{\omega^2+b^2} \times \overline{\omega^2+c^2}$, &c.

Whence, by extracting the ſquare root of both ſides of thoſe equations, we have, when n is an odd number, $\overline{a+\omega}^{n} + \overline{a-\omega}^{n} = 2an \times \sqrt{\omega^2+b^2} \times \sqrt{\omega^2+c^2}$, &c. $2a$ taking place inſtead of $\sqrt{\omega^2 + \text{ſq. of the tang. of } 90^\circ}$: And, when n is an even number, $\overline{a+\omega}^{n} + \overline{a-\omega}^{n} = 2 \times \sqrt{\omega^2+b^2} \times \sqrt{\omega^2+c^2}$, &c. Hence we infer this conſtruction.

XIII.

Having deſcrib'd about the centre C (*fig.* 3. and 4.), with the radius a, the circle $P\,a'\,A'\,a''\,A''$, &c. draw the diameter $P\,C\,Q$, and the tangent $b''\,P\,b^4$; divide the ſemicircumference $P\,a'\,Q$ into as many equal parts $P\,a'$, $a'\,A'$, $A'\,a''$, &c. as there are units in $2n$; draw the ſecants $C\,a'\,b'$, $C\,a''\,b''$, &c. and, thro' any point (O) in $C\,Q$, draw $k''\,O\,k^4$ parallel to $b''\,P\,b^4$; likewiſe draw $b'\,k'$, $b''\,k''$, &c. parallel to $P\,Q$; and call $C\,O$, ω.

Then, if the radius be 1, p will be the coſine of twice the angſe $P\,C\,a'$, q the coſine of twice $P\,C\,a''$, &c. therefore $P\,b' = O\,k'$ will be $= b$, $P\,b'' = O\,k''$ $= c$, &c. and $C\,k' = \sqrt{\omega^2+b^2}$, $C\,k'' = \sqrt{\omega^2+c^2}$, &c.

Con-

Conſequently $OP^n + OQ^n$ being $= \overline{a+\omega}^n + \overline{a-\omega}^n$, and $n \times PQ \times Ck' \times Ck''$, &c. $= 2an \times \sqrt{\overline{\omega^2 + b^2}} \times \sqrt{\overline{\omega^2 + c^2}}$, &c. when n is an odd number; $OP^n + OQ^n$ will then be $= n \times PQ \times Ck' \times Ck''$, &c. where the diameter PQ takes place inſtead of the infinite quantity $C\overset{\frac{n+1}{2}}{k}$.

But if n be an even number, $OP^n + OQ^n$ will be $= 2 \times Ck' \times Ck''$, &c.

XIV.

It is obvious that, of the factors Ck', Ck'', &c. the firſt and laſt, the ſecond and laſt but one, *&c.* are reſpectively equal to each other: Therefore the ſquares of the factors below PQ, and the ſquares of their values, being omitted,

$OP^n + OQ^n$ is $= n \times PQ \times Ck'^2 \times Ck''^2$, &c. and

$\overline{a+\omega}^n + \overline{a-\omega}^n = 2an \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$, &c. when n is an odd number; or

$OP^n + OQ^n$ is $= 2 \times Ck'^2 \times Ck''^2$, &c. and

$\overline{a+\omega}^n + \overline{a-\omega}^n = 2 \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$, &c. when n is an even number.

XV.

Writing in the equation $\overline{a+\omega}^{2n} - 2y \times \overline{a+\omega \times a-\omega}^{n} + \overline{a-\omega}^{2n} = \overline{2 \mp 2y} \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$, &c.

(found by *Art*. VII.) $a - u$ for ω, the ſame becomes $\overline{2a - u}^{2n} - 2yu^{n} \times \overline{2a - u}^{n} + u^{2n} = \overline{2 \mp 2y} \times \overline{u^2 - 2au + a^2 + b^2} \times \overline{u^2 - 2au + a^2 + c^2}$, &c. $= \overline{2 \mp 2y} \times \overline{u^2 - 2au + \beta^2} \times \overline{u^2 - 2au + \gamma^2} \times \overline{u^2 - 2au + \delta^2}$, &c. if inſtead of $\sqrt{a^2 + b^2}$, $\sqrt{a^2 + c^2}$, &c. (the ſecants of the arcs of which b, c, d, &c. are tangents), we put β, γ, δ, &c.

And, by a like ſubſtitution in the equations in *Art*. XI. and XIV. it appears, that $\overline{2a - u}^{n} - u^{n}$ is $= 2an \times \overline{a - u} \times \overline{u^2 - 2au + \gamma^2} \times \overline{u^2 - 2au + \delta^2}$, &c. or $2 \times \overline{a - u} \times \overline{u^2 - 2au + \gamma^2} \times \overline{u^2 - 2au + \delta^2}$, &c. according as n is an even or an odd number: And that $\overline{2a - u}^{n} + u^{n}$ is $= 2an \times \overline{u^2 - 2au + \beta^2} \times \overline{n^2 - 2au + \gamma^2}$, &c. or $2 \times \overline{u^2 - 2au + \beta^2} \times \overline{u^2 - 2au + \gamma^2}$, &c. according as n is an odd or an even number.

From what is done above, I might now deduce many corollaries; and, by means of other ſubſtitutions, inveſtigate other theorems; but want of leiſure obliges me to deſiſt.

LXXVII. *An*

Fig. 1.

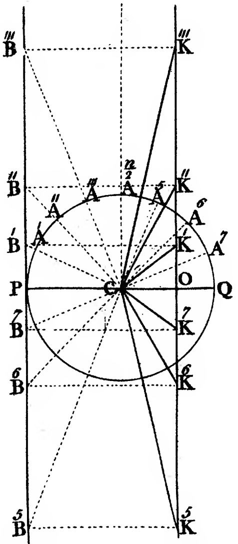


Fig. 3.

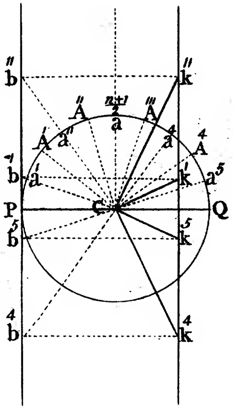


Fig. 2.

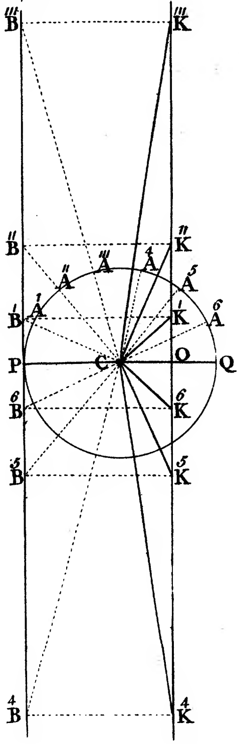
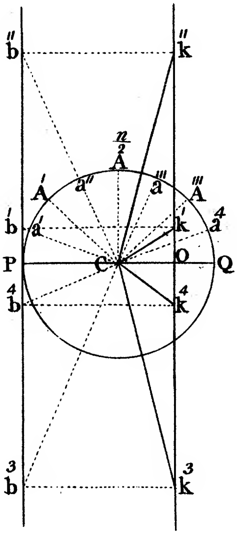


Fig. 4.



J. Mynde fc.